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Decay of solutions of the wave equation with arbitrary localized nonlinear damping

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Abstract

We study the problem of decay rate for the solutions of the initial–boundary value problem to the wave equation, governed by localized nonlinear dissipation and without any assumption on the dynamics (i.e., the control geometric condition is not satisfied). We treat separately the autonomous and the non-autonomous cases. Providing regular initial data, without any assumption on an observation subdomain, we prove that the energy decays at last, as fast as the logarithm of time. Our result is a generalization of Lebeau (in: A. Boutet de Monvel, V. Marchenko (Eds.), Algebraic and Geometric Methods in Mathematical Physics, Kluwer Academic Publishers, Dordrecht, the Netherlands, 1996, pp. 73) result in the autonomous case and Nakao (Adv. Math. Sci. Appl. 7 (1) (1997) 317) work in the non-autonomous case. In order to prove that result we use a new method based on the Fourier–Bross–Iaglintzer (FBI) transform.

Keywords: Decay rate; Initial–boundary value problem; Wave equation; FBI transform

1. Introduction

The aim of this paper is to give decay estimates for the wave equation with a nonlinear damping term and without any assumption on the dynamics. Let Ω be a smooth n-dimensional Riemannian compact manifold with boundary $\Gamma = \partial\Omega$. We start an investigation of the asymptotic behavior of the solution of the following

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wave equation:

$$\partial_t^2 u - \Delta u + b(t,x) g(\partial_t u) = 0 \quad \text{in} \quad Q = \Omega \times \mathbb{R}^+, \tag{1.1}$$

with the Dirichlet boundary condition:

$$u = 0 \quad \text{on} \quad \Sigma = \Gamma \times \mathbb{R}^+, \tag{1.2}$$

where $b(t,x)$ is given by

$$b(t,x) = (1+t)^\theta a(x) := \sigma(t) a(x), \quad -1 < \theta \leqslant 0, \tag{1.3}$$

Δ is the Laplace–Beltrami operator on Ω, $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a non-decreasing continuous function with $g(0) = 0$, $sg(s) \geqslant 0$ and $a \in L^\infty(\Omega)$ is assumed to be a positive function $a(x) \geqslant 0$ for all $x \in \Omega$. Let $\omega \subset\subset \Omega$ be a given arbitrary non-empty subdomain such that $a(x) \geqslant a_0 > 0$ in $\omega \subset\subset \Omega$. We are interested in a *semi-dynamical system* associated with (1.1) and (1.2). Let us take the product-space $X = H_0^1(\Omega) \oplus L^2(\Omega)$, where the norm in $H_0^1(\Omega)$ is defined by

$$\|v\|_{H_0^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)}, \quad v \in H_0^1(\Omega). \tag{1.4}$$

The norm in X is chosen as follows:

$$\|(v,w)\|_X^2 = E(v,w) = \|v\|^2_{H_0^1(\Omega)} + \|w\|^2_{L^2(\Omega)}, \quad \text{for} \quad (v,w) \in X. \tag{1.5}$$

It is known that (1.1) and (1.2) define an evolution in X in a natural way: any initial state $u = (u_0, u_1) \in X$ will transform in time into the state $(u(t), \partial_t u(t))$, with the initial conditions

$$u(0,x) = u_0, \quad \partial_t u(0,x) = u_1. \tag{1.6}$$

Thus, from the very beginning we have to impose certain restrictions on g in order to guarantee the global existence, uniqueness and continuous dependence on the initial data. We will assume that $g(s)$ satisfies the following conditions:

(i) There exists $C_1, C_2 > 0$ and $r \geqslant 1$ such that for $|s| \leqslant 1$, we have

$$C_1 |s|^r \leqslant |g(s)| \leqslant C_2 |s|^{\frac{1}{r}}. \tag{1.7}$$

(ii) There exists $C_1', C_2' > 0$ such that for $|s| > 1$ we have

$$C_1' |s|^k \leqslant |g(s)| \leqslant C_2' |s|^p,$$

where

$$0 \leqslant k \leqslant 1, \quad 1 \leqslant p < \frac{n}{n-2}, \quad (n-2)(1-k) \leqslant 4r. \tag{1.8}$$

1.1. The autonomous case

The primary consideration of this paper is the decay rate of solutions of (1.1) and (1.2) when $\theta = 0$.

It is well known that, due to the monotonicity of the nonlinear term, the forward initial value problem

$$\begin{cases} \partial_t^2 u - \Delta u + a(x)g(\partial_t u) = 0 & \text{in} \quad Q = \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on} \quad \Sigma = \Gamma \times \mathbb{R}^+, \\ u(x,0) = u_0, \quad \partial_t u(x,0) = u_1 & \text{in} \quad \Omega \end{cases} \tag{1.9}$$

can always be solved for $(u_0, u_1) \in X$ yielding a unique global bounded solution in X (e.g. [11,15,16,27]). We define the evolution operators by

$$S(t,.) : \begin{cases} X & \longrightarrow X, \\ (u_0, u_1) & \longrightarrow (u(t), \partial_t u(t)), \end{cases} \tag{1.10}$$

where $u \in \mathcal{C}(\mathbb{R}^+, H_0^1(\Omega)) \cap \mathcal{C}^1(\mathbb{R}^+, L^2(\Omega))$ is the unique solution of (1.9) with initial data (u_0, u_1). Let us consider the energy at instant t defined by

$$E(u,t) = \frac{1}{2} \int_\Omega \left|\partial_t u(t,x)\right|^2 + |\nabla u(t,x)|^2 \, dx, \tag{1.11}$$

where dx is the Riemannian volume element on Ω.

For any solution $u(t)$ of (1.9), the function $E(u,t)$ satisfies the following identity:

$$E(u,t) - E(u,0) = -\int_0^t \int_\Omega a(x)g(\partial_t u)\partial_t u \, dx \, ds, \tag{1.12}$$

and therefore the energy is a non-increasing function of time t. Moreover, when $g(s)s > 0$ for $s \neq 0$, the trajectories $(u(t), \partial_t u(t))$ are shown to converge in X to the single stationary state of the system, i.e. $(0,0)$. This can be achieved with the help of Lasalle's invariance principle by imposing conditions ensuring the existence of a strict Liapunov function and the precompactness of trajectories. However, this method does not yield explicit estimates of the decay rate (see [7], for instance, and the references therein).

1.2. Comments on the existing papers

In recent years several works on this subject have appeared, mainly concerned with the asymptotic behavior when $g(s)$ grows as a power of s. Moreover, some decay estimates of the energy are proved only when the dissipation mechanism is effective in the neighborhood of a suitable subset of the boundary.

The results of Dafermos [6] and Haraux [7], based on Lasalle invariance principle, show that the energy of every solution goes to zero as t goes to infinity. In [6], Dafermos studies the phenomenon of "*stabilization*" of trajectories for the wave equation in a bounded open domain with a weak dissipative mechanism $a \in L^\infty(\Omega)$, $a(x) \geqslant 0$ a.e. in Ω. He showed that if $meas(supp(a)) > 0$ and g is continuously differentiable, strictly increasing function in $\mathbb{R}$, then the energy of any weak solution tends to zero as t tends to infinity. Haraux in [7] generalized Dafermos' result to a framework which allows the existence of non-trivial equilibria and where g is neither strictly increasing nor smooth but is just a maximal monotone graph. In [25], Slemord generalized Haraux's results and removed the assumption of monotonicity. However, Slemord proved that if $meas(supp(a)) > 0$, then any weak solution of (1.9) converges weakly to zero, provided g is globally Lipschitz.

Following a recent work of Vancostenoble [28] investigates the weak stabilization to zero of a solution to Eqs. (1.9). The term $a(x)g(\partial_t u)$ represents a possibly non-monotone feedback dissipation on a "non-negligible" part ω of $\overline{\Omega}$.

Therefore, it is interesting to investigate whether the energy of the solution of problem (1.9) converges to zero. We say that problem (1.9) has uniform energy decay property when there exists a continuous function $f(t)$ tending to zero as t tends to infinity such that

$$E(u,t) \leqslant C(u_0, u_1) f(t) \tag{1.13}$$

where a constant $C(u_0, u_1)$ depends upon the initial energy in a bounded way.

In this case the function $g(s)$ (like $|s|^\sigma s$ with $\sigma > -1$) and ω satisfy the geometrical control condition (i.e. there exists some $T > 0$ such that every ray of geometric optics intersects the set $\omega \times (0, T)$. The canonical example of the open subset ω verifying the "control geometric condition" is when ω is a neighborhood of the boundary. Nakao [17] proves that we can take $f(t)$ as $(1+t)^{-\gamma}$ for some $\gamma > 0$ and provide a regular data $(u_0, u_1) \in (H^2 \cap H_0^1) \oplus H_0^1$. Posteriorly, Chentouh [5] established the same result for $g(s) = c_1 |s|^{p-1} s + c_2 |s|^{q-1} s$, where $c_1, c_2 > 0$, $1 \leqslant q \leqslant p$, $(n-2)q \leqslant n+2$ with the additional conditions $q \leqslant 2(p+1)$, $((n-2)(p+2) + 2n)q \leqslant (n+2)(p+4)$. In Haraux-Zuazua [9] and Zuazua [29] these restrictions on q were removed by using the same Liapunov approach. Subsequently, Tébou [27] generalized Nakao results to include a large class of functions g, but the damping term is effective in a neighborhood of a suitable subset of the boundary. In other words, the geometrical control condition is fulfilled. The method of proof of [27] is based on the multipliers technique.

In the present paper we show that even if the geometrical condition is not fulfilled (or the damping is effective in an arbitrary domain), then the energy decays with respect to time at last as fast as the inverse of the logarithm. Moreover, our main goals here are finding how the constant $C(u_0, u_1)$ is affected by the nonlinearity.

In order to prove that result we use a new method based on the Carleman estimates and the Fourier–Bros–Iaglontzer (FBI) transformation for the linear part of the solution.

The linear case, i.e. $g(s) = s$ and without any assumption on the dynamics, has been treated by Lebeau [12] and Lebeau-Robbiano [14]. Their his proofs are essentially based on the spectral calculus. In order to prove that result, he bound the spectrum of A from below. This bound is obtained by using a Carleman-type estimate for the resolvent.

Unfortunately, this method does not seem to extend to the initial–boundary problems with a nonlinear dissipation mechanism, even when the nonlinearity g commutes with the phase, that is $g(e^{i\lambda}v) = e^{i\lambda}g(v)$, for all $\lambda \in \mathbb{R}$.

1.3. Statement of main results

Our main results are the decay rate for the solutions of the initial–boundary problem to the wave equation, and the main achievements of this paper are

- arbitrariness of the observation subdomain ω,
- the improvement of the constant in the decay rate for our problem.

Let us take the product-space $X^1(\Omega) = (H^2(\Omega) \cap H_0^1(\Omega)) \oplus H_0^1(\Omega)$ as the state space of our system. The norm in $X^1(\Omega)$ is chosen as follows:

$$\|(u_0, u_1)\|^2_{X^1(\Omega)} = \left\|\Delta u_0\right\|^2 + \|\nabla u_1\|^2, \quad \text{for any} \quad (u_0, u_1) \in X^1(\Omega). \tag{1.14}$$

Before stating the main results, we recall the following lemma on the unique existence of a strong solution to problem (1.1), which we shall use repeatedly in the sequel. The proof is based on [16]. We can also refer to [15,27].

Theorem 1. *Let $(u_0, u_1) \in X^1(\Omega)$ and $a \in L^\infty(\Omega)$. Assume that the continuous, non-decreasing function g satisfies (1.7) and (1.8). Then the solution u of (1.9) starting from (u_0, u_1) satisfies*

$$u \in L^\infty(\mathbb{R}^+; H^2(\Omega) \cap H_0^1(\Omega)) \cap W^{1,\infty}(\mathbb{R}^+; H_0^1(\Omega)) \cap W^{2,\infty}(\mathbb{R}^+, L^2(\Omega)). \tag{1.15}$$

Moreover, there exists a positive constant C such that

$$\left\|\nabla \partial_t u(t)\right\|_2^2 + \left\|\partial_t^2 u(t)\right\|_2^2 \leqslant C \,\|(u_0, u_1)\|^2_{X^1(\Omega)}, \quad \textit{for a.e.} \quad t \geqslant 0. \tag{1.16}$$

The first main result of this paper can be stated as follows:

Theorem 2. *Let $(u_0, u_1) \in X^1(\Omega)$. Assume that the continuous non-decreasing function g satisfies (1.7) and (1.8). Then there exists a positive constant $C > 0$ such that* *the estimate*

$$E(u,t)\leqslant C\Phi\big(\,\|(u_0,u_1)\|^2_{X^1}\big)\big(\log(2+t)\big)^{-1} \tag{1.17}$$

holds for a solution of (1.9) *starting from* (u_0,u_1), *where the real function* Φ *is given by*

$$\Phi(z)=\begin{cases}1+z+z^{1+\frac{r-k}{r+1}} & \text{if} \quad p=1,\\ 1+z+z^{1+\frac{r-k}{r+1}}+z^{p+\frac{\sigma}{2}} & \text{if} \quad p>1,\end{cases} \tag{1.18}$$

where $\sigma>0$ *is a fixed real that satisfies*: $p+\frac{\sigma}{2}\leqslant\frac{n}{n-2}$.

We now give some example of application of Theorem 2.
We consider the autonomous problem

$$\begin{cases}\partial_t^2 u-\Delta u+a(x)g(\partial_t u)=h(x) & \text{in} \quad Q=\Omega\times\mathbb{R}^+,\\ u=0 & \text{on} \quad \Sigma=\Gamma\times\mathbb{R}^+,\\ u(x,0)=u_0, \quad \partial_t u(x,0)=u_1 & \text{in} \quad \Omega.\end{cases} \tag{1.19}$$

Problem (1.19) has been studied by Amerio and Prouse [2], Brezis [4], Lions and Strauss [16], Haraux [8], and Zuazua [29]. A particular solution of this problem is the unique equilibrium solution u^* that satisfies the following problem:

$$\begin{cases}-\Delta u^*=h(x) & \text{in} \quad \Omega,\\ u^*=0 & \text{on} \quad \Gamma.\end{cases} \tag{1.20}$$

We now apply Theorem 2 in order to obtain decay rate of the solution $u(t)$ to u^* in the space $X(\Omega)$. In order to include this model in the framework of Theorem 2, it is enough to take $v=u-u^*$. Then v satisfies

$$\begin{cases}\partial_t^2 v-\Delta v+a(x)g(\partial_t v)=0 & \text{in} \quad Q=\Omega\times\mathbb{R}^+,\\ v=0 & \text{on} \quad \Sigma=\Gamma\times\mathbb{R}^+.\end{cases} \tag{1.21}$$

Then, as immediate consequence of our main result, the following conclusion holds for system (1.19).

Corollary 1.1. *Let* $(u_0,u_1)\in X^1(\Omega)$. *Assume that the continuous, non-decreasing function g satisfies* (1.7) *and* (1.8). *Then for any solution u of* (1.19) *starting from* (u_0,u_1)

the following estimate holds:

$$\left\|u(t)-u^*\right\|_{H_0^1}^2+\left\|\partial_t u(t)\right\|_{L^2}^2 \leqslant \frac{C}{\log(2+t)}. \tag{1.22}$$

Here the constant C is dependent on the initial data of u.

1.4. The non-autonomous case

A natural question is whether the result described in the previous section applies to non-autonomous evolution equations as well. Here we shall investigate the decay property of the solutions to the initial–boundary value problem for the wave equation with a local time-dependent nonlinear damping. More specifically, we consider the following problem:

$$\begin{cases} \partial_t^2 u-\Delta u+b(t,x)g(\partial_t u)=0 & \text{in} \quad Q=\Omega\times\mathbb{R}^+, \\ u=0 & \text{on} \quad \Sigma=\Gamma\times\mathbb{R}^+, \\ u(x,0)=u_0, \quad \partial_t u(x,0)=u_1 & \text{in} \quad \Omega. \end{cases} \tag{1.23}$$

In the case of the wave equation with time-dependent dissipations there are not many results which give decay rates of solutions. It is shown, in particular, that the effect of the time dependence on the decay rate is very delicate. For some class of functions g, Nakao [18] proves a precise decay estimate of the solutions of the initial–boundary value problem (1.23) for a regular data $(u_0,u_1)\in(H^2\cap H_0^1)\oplus H_0^1$, provided that the geometric control condition is satisfied. His results are a generalization of [17], where the nonlinear dissipation treated is independent of time.

An important special case of (1.23) occurs when $g(s)=s$ and $a(x)=1$, i.e. the damping term is effective everywhere in the domain Ω, so that (1.23) takes the form

$$\partial_t^2 u-\Delta u+(1+t)^\theta\partial_t u=0. \tag{1.24}$$

The behavior of solutions as $t\to\infty$ depends crucially on the parameter θ. We can show that if $\left|\theta\right|\leqslant 1$, then the rest field is asymptotically stable. On the other hand, when $\theta<-1$ there exist oscillatory solutions that do not approach zero when $t\to\infty$.

We suppose as before that $-1<\theta<0$ and condition (1.8) is satisfied with $p=1$.

The second main result of this paper may be stated as follows:

Theorem 3. *Let $(u_0,u_1)\in X^1(\Omega)$. Then, under assumptions* (1.7) *and* (1.8) *problem* (1.23) *admits a unique solution u such that*

$$u\in L^\infty(\mathbb{R}^+;H^2(\Omega)\cap H_0^1(\Omega))\cap W^{1,\infty}(\mathbb{R}^+;H_0^1(\Omega))\cap W^{2,\infty}(\mathbb{R}^+,L^2(\Omega)) \tag{1.25}$$

and the decay property

$$E(u,t) \leqslant C\Phi\left(\|(u_0,u_1)\|_{X^1}^2 \right)\left(\log(2+t)\right)^{-1} \tag{1.26}$$

holds for a solution of (1.23) *starting from* (u_0, u_1).

1.5. Some remarks

1. Let us observe that a function g satisfying (1.7) and (1.8) is not necessary globally Lipschitz. In fact, the class of functions satisfying (1.7) and (1.8) includes functions like $s\,|s|^{r-1}$, $r > 1$. We also observe that bounded function g can be used.
2. One should note that the unique continuation principle is much weaker than the geometric control hypothesis (see [3]). In particular, one can show ([20,23,26]) that for any connected Ω and any ω of non-zero measure, there exists a time T such that $\omega\times]0,T[$ satisfies the unique continuation principle, i.e. any solution of the problem

$$\partial_t^2 u - \Delta u = 0, \quad u_{|\partial\Omega\times\mathbb{R}^+} = 0, \quad u_{|\omega\times[0,T]} = 0 \tag{1.27}$$

   is identically zero. In particular, Theorem 2 just means that if one waits long enough and if one observes phenomena which have some regularity, then we have decay property of the energy, even in the absence of the geometrical property.
3. In the case when $\omega = \Omega$, i.e. the damping term is effective everywhere in Ω, Salvator and Vitillaro [24] gives some decay rate of the solutions in some particular case of (1.7) and (1.8).
4. For systems of second-order ordinary differential equations with time-dependent dissipation, some general conditions for the solutions to decay to zero are already known (see Pucci and Serrin [19] and the references cited there). Even for ordinary differential equations, however, there are few results which further give decay rates for the solutions.

We treat separately the autonomous case and the non-autonomous case, for which slightly different types of hypothesis and estimates are required.

The remainder of this paper is organized as follows. In Section 2, we give some lemmas which are used for the proof of the main results. In Section 3, we prove Theorem 2. Section 4 is devoted to the proof of the weak observability inequality, and finally, in Section 5 we prove Theorem 3.

Throughout this paper, we denote by $\|v\|_p$ the norm of a function $v \in L^p(\Omega)$, $1 \leqslant p \leqslant \infty$. We use the following additional notations:

$$\Omega_1 = \{x \in \Omega;\ \left|\partial_t u(t,x)\right| \leqslant 1\}, \quad \Omega_2 = \{x \in \Omega;\ \left|\partial_t u(t,x)\right| > 1\} \tag{1.28}$$

for a fixed $t \geqslant 0$.

2. Some preliminary lemmas

In this section we first derive several preliminary estimates which are the starting point of the proof of Theorems 2 and 3. We shall use the following notations. The evolution operator $S(t,.)$ defined by (1.10) can be written as a sum

$$S(t,.) = V(t,.) + W(t,.), \tag{2.1}$$

where $V(t,.) : X \longrightarrow X$ is a continuous operator defined as the solution operator of the linear problem:

$$\begin{cases} \partial_t^2 v - \Delta v = 0 & \text{in} \quad Q = \Omega \times \mathbb{R}^+, \\ v = 0 & \text{on} \quad \Sigma = \Gamma \times \mathbb{R}^+, \\ v(x,0) = u_0, \quad \partial_t v(x,0) = u_1 & \text{in} \quad \Omega, \end{cases} \tag{2.2}$$

and $W_t(u_0,u_1) = (w(t), \partial_t w(t))$, where $w(t)$ is the solution of the inhomogeneous wave equation with zero Cauchy data at $t=0$:

$$\begin{cases} \partial_t^2 w - \Delta w = -a(x)g(\partial_t u) & \text{in} \quad Q = \Omega \times \mathbb{R}^+, \\ w = 0 & \text{on} \quad \Sigma = \Gamma \times \mathbb{R}^+, \\ w(x,0) = 0, \quad \partial_t w(x,0) = 0 & \text{in} \quad \Omega. \end{cases} \tag{2.3}$$

We will begin with a fundamental estimate for the linear part of the solution (a weak observation) which represents a quantitative form of the unique continuation principle for the solution of (2.2).

Proposition 2.1. *There exist constants $T, \mu, C > 0$ and $\lambda_0 > 0$ such that for any $\lambda \geqslant \lambda_0$ the following estimate*

$$E(u_0,u_1) \leqslant C\left[\frac{1}{\lambda}\,\|(u_0,u_1)\|^2_{X^1(\Omega)} + e^{\mu\lambda}\int_0^T\int_\Omega a(x)\left|\partial_t v(s,x)\right|^2\,dx\,ds\right] \tag{2.4}$$

holds for the solution, $v \in H_0^1(\Omega \times [0,T])$, of problem (2.2) starting from $(u_0,u_1) \in X^1(\Omega)$.

To prove Proposition 2.1, we use the idea of Robbiano [20,21] to apply the Fourier–Bros–Iagolnitzer transformation and the proof is given in Section 4.

Remark 1. Estimate (2.4) proves in particular the unique continuation property, i.e., if $a(x)\partial_t v = 0$ for $t \in [0,T]$, then the solution v is identically zero (see [10,26]).

Lemma 2.1. *Let u be the solution of system (1.9) starting from $(u_0, u_1) \in X^1$. Then there exists constants $T, C, \mu > 0$ and $\lambda_0 > 0$ such that for any $\lambda \geqslant \lambda_0$ the following estimate*

$$E(u, t+T) \leqslant C\Big(\frac{1}{\lambda}\Phi(\|(u_0, u_1)\|^2_{X^1(\Omega)}) + e^{\mu\lambda}\int_t^{t+T}\int_\Omega a(x)\left|\partial_t u(s,x)\right|^2 + \left|a(x)g(\partial_t u)\right|^2\,dx\,ds\Big) \tag{2.5}$$

holds for any $t \geqslant 0$.

Proof. We write the solution $u(t,x)$ of (1.9) as

$$u(t,x) = v(t,x) + w(t,x), \tag{2.6}$$

where $v(t,x)$ solves (2.2) and $w(t,x)$ satisfies (2.3).

From (2.4) and using the non-increasing character of the energy, we obtain

$$E(u,T) \leqslant C\left[\frac{1}{\lambda}\|(u_0,u_1)\|^2_{X^1(\Omega)} + e^{\mu\lambda}\Big(\int_0^T\int_\Omega a(x)\left|\partial_t u(s,x)\right|^2\,dx\,ds + \|w\|^2_{H^1((0,T)\times\Omega)}\Big)\right]. \tag{2.7}$$

Applying the energy estimate to (2.3), we obtain

$$\begin{aligned}\|w\|^2_{H^1((0,T)\times\Omega)} &\leqslant C\left\|ag(\partial_t u)\right\|^2_{L^1((0,T);L^2(\Omega))}\\ &\leqslant C\left\|ag(\partial_t u)\right\|^2_{L^2((0,T)\times\Omega)}.\end{aligned} \tag{2.8}$$

Collecting (2.8) and (2.7), we deduce that

$$E(u,T) \leqslant C\Big(\frac{1}{\lambda}\|(u_0,u_1)\|^2_{X^1(\Omega)} + e^{\mu\lambda}\int_0^T\int_\Omega a(x)\left|\partial_t u(s,x)\right|^2 + \left|a(x)g(\partial_t u)\right|^2\,dx\,ds\Big). \tag{2.9}$$

Applying (2.9) to $S(t)(u_0, u_1)$, we obtain

$$E(u,t+T) \leqslant C\Big(\frac{1}{\lambda}\|S(t)(u_0,u_1)\|^2_{X^1(\Omega)} + e^{\mu\lambda}\int_t^{t+T}\int_\Omega a(x)\left|\partial_t u(s,x)\right|^2 + \left|a(x)g(\partial_t u)\right|^2\,dx\,ds\Big). \tag{2.10}$$

However, using (1.16)–(1.8), we find

$$\begin{aligned}\|S(t)(u_0,u_1)\|^2_{X^1} &= \left\|\Delta u(t)\right\|^2 + \left\|\nabla\partial_t u(t)\right\|^2\\ &\leqslant C\left[\left\|\nabla\partial_t u(t)\right\|^2 + \left\|\partial_t^2 u(t)\right\|^2 + \left\|ag(\partial_t u)\right\|^2\right]\\ &\leqslant C\left[\|(u_0,u_1)\|^2_{X^1} + \int_{\Omega_1}\left|\partial_t u(t)\right|^{2/r}\,dx + \int_{\Omega_2}\left|\partial_t u(t)\right|^{2p}\,dx\right]\\ &\leqslant C\left[\|(u_0,u_1)\|^2_{X^1} + \left(\int_{\Omega}\left|\partial_t u(t)\right|^2\,dx\right)^{\frac{1}{r}}\right.\\ &\qquad\left. + \int_{\Omega}\left|\partial_t u(t)\right|^{2p}\,dx\right]. \end{aligned} \tag{2.11}$$

By Sobolev's imbedding theorem (e.g., Adams [1] and Ikawa [11]) and (1.16), we have

$$\|S(t)(u_0,u_1)\|^2_{X^1} \leqslant C\Phi(\|(u_0,u_1)\|^2_{X^1}), \tag{2.12}$$

where Φ is defined in the same way as above. The proof of Lemma 2.1 is complete. □

In the proof of Theorems 2 and 3 we will find some differential inequalities. In the following technical lemma, we solve these inequalities and we deduce some decay estimates which will be used in the proof of the main results. We will get the following lemma:

Lemma 2.2. *Let $(a_n)_n$ be a decreasing sequence of real numbers such that $0\leqslant a_n\leqslant 1$, and assume that, there exists $\lambda_0>0$ such that for any $\lambda\geqslant\lambda_0$ we have*

$$a_{n+1}\leqslant \frac{C_0}{\lambda} + n^{\eta_1}e^{\mu\lambda}\Big(a_n - a_{n+1}\Big)^{\varepsilon_1} + n^{\eta_2}e^{\mu\lambda}\Big(a_n - a_{n+1}\Big)^{\varepsilon_2} \tag{2.13}$$

for some

$$\eta_1 < \varepsilon_1 \leqslant 1, \quad \eta_2 < \varepsilon_2 \leqslant 1. \tag{2.14}$$

Then there exists a constant $C>0$ such that

$$a_n \leqslant \frac{C}{\log(n)}, \quad \forall n \geqslant 2. \tag{2.15}$$

Proof. Let $\beta < \min\left\{\varepsilon_1 - \eta_1, \varepsilon_2 - \eta_2\right\}$. Selecting

$$\lambda = \frac{\beta}{\mu} \log\left(\frac{1}{a_n - a_{n+1}} + 2\right) \tag{2.16}$$

we claim that

$$\lim_{n\to\infty} (a_n - a_{n+1}) = 0. \tag{2.17}$$

Indeed the sequence $(a_n)_n$ is non-increasing and bounded from below; its limit exists. Now by (2.13), we have

$$a_{n+1} \leqslant C_0 \log\left(\frac{1}{a_n - a_{n+1}} + 2\right)^{-1} + n^{\eta_1}(a_n - a_{n+1})^{\varepsilon_1 - \beta} + n^{\eta_2}(a_n - a_{n+1})^{\varepsilon_2 - \beta}. \tag{2.18}$$

Setting $\gamma_n = \log(n)a_n$, we obtain $\gamma_n \leqslant \log(n)$. Since

$$\frac{1}{\log(n)} - \frac{1}{\log(n+1)} = \int_n^{n+1} \frac{du}{u(\log(u))^2} \leqslant \frac{1}{n(\log(n))^2}, \tag{2.19}$$

we have

$$\gamma_n\left(\frac{\log(n+1)}{\log(n)} - 1\right) \leqslant \frac{\log(n+1)}{n(\log(n))} \leqslant \frac{\log(n+1)}{n}, \quad \forall n \geqslant n_0. \tag{2.20}$$

We distinguish two cases:

(i) If $a_n - a_{n+1} \leqslant \frac{1}{n}$, by (2.18), we obtain

$$\gamma_{n+1} \leqslant C_0. \tag{2.21}$$

(ii) If $a_n - a_{n+1} \geqslant \frac{1}{n}$, using (2.20), we obtain

$$\gamma_{n+1} \leqslant \gamma_n\left(\frac{\log(n+1)}{\log(n)}\right) - \frac{\log(n+1)}{n} \leqslant \gamma_n, \quad \forall n \geqslant n_0. \tag{2.22}$$

Combining (2.21) and (2.22), we obtain

$$\forall n \geqslant n_0, \quad \gamma_{n+1} \leqslant \max(C_0, \gamma_n). \tag{2.23}$$

Setting $C_1 = \max(C_0, \log(n_0))$, we obtain

$$a_n \leqslant C_1(\log(n))^{-1}, \quad \forall n \geqslant 2. \tag{2.24}$$

This completes the proof. $\square$

3. Proof of Theorem 2

This section is devoted to the proof of Theorem 2.

3.1. Some preliminary estimates

First of all we estimate the last term of inequality (2.5).

Lemma 3.1. *There exists $C, \gamma > 0$, such that for any $\varepsilon > 0$ the following estimate*

$$\int_t^{t+T}\int_{\Omega_2} \left|a(x)g(\partial_t u)\right|^2 \, dx \, ds \;\leqslant\; \varepsilon\Phi(\|(u_0,u_1)\|^2_{X^1}) + C\varepsilon^{-\gamma}\left[\int_t^{t+T}\int_{\Omega} a(x)g(\partial_t u)\partial_t u \, dx \, ds\right]^{\frac{2}{r+1}} \tag{3.1}$$

holds, for any $t \geqslant 0$, and Φ given by (1.18).

Proof. By an interpolation inequality we have the following estimate:

$$\left\|a^{1/p}\partial_t u(s)\right\|_{2p} \leqslant \left\|a^{1/p}\partial_t u(s)\right\|_2^{1-\eta} \left\|\partial_t u(s)\right\|^{\eta}_{2p+\sigma}, \tag{3.2}$$

where

$$\frac{1}{2p} = \frac{1-\eta}{2} + \frac{\eta}{2p+\sigma} \tag{3.3}$$

and then η is given by

$$\eta = 1 - \frac{\sigma}{p(2p-2+\sigma)} < 1. \tag{3.4}$$

Since

$$ab \leqslant \frac{a^{k_0}}{k_0\varepsilon^{k_0}} + \frac{b^{k_0'}\varepsilon^{k_0'}}{k_0'} \tag{3.5}$$

with

$$k_0 = \frac{1}{(1-\eta)p}, \quad \frac{1}{k_0} + \frac{1}{k_0'} = 1 \tag{3.6}$$

we obtain the following estimate:

$$\int_{\Omega_2} a^2(x) \left|\partial_t u(s)\right|^{2p} \, dx \leqslant C\varepsilon^{-k_0} \int_{\Omega_2} a^{2/p} \left|\partial_t u(s)\right|^2 \, dx + \varepsilon^{k_0'} \int_{\Omega} \left|\partial_t u(s)\right|^{2p+\sigma} \, dx. \tag{3.7}$$

We will assume that $p + \frac{\sigma}{2} \leqslant \frac{n}{n-2}$. Then by Sobolev's imbedding theorem, we get

$$\left\|\partial_t u(s)\right\|_{2p+\sigma} \leqslant \left\|\nabla \partial_t u(s)\right\|_2 \leqslant C \left\|(u_0, u_1)\right\|_{X^1(\Omega)}. \tag{3.8}$$

On the other hand, using the growth condition imposed on g by assumption (1.8), we obtain the following estimate, for $\delta = \frac{1}{r+1}$:

$$\int_{\Omega_2} a^{2/p} \left|\partial_t u(s)\right|^2 \, dx \leqslant C \int_{\Omega} a^{(2/p)-\delta} \left|\partial_t u(s)\right|^{2-\delta(k+1)} (a(x) g(\partial_t u) \partial_t u)^{\delta} \, dx. \tag{3.9}$$

Applying Hölder's inequality, we obtain

$$\begin{aligned}\int_{\Omega_2} a^{2/p} \left|\partial_t u(s)\right|^2 \, dx &\leqslant C \left[\int_{\Omega_2} a^{((2/p)-\delta)/1-\delta} \left|\partial_t u(s)\right|^{\frac{2-\delta(k+1)}{1-\delta}} \, dx\right]^{1-\delta} \\ &\quad \times \left[\int_{\Omega} a(x) g(\partial_t u) \partial_t u \, dx\right]^{\delta} \\ &\leqslant C\varepsilon^{2k_0} \left[\int_{\Omega} \left|\partial_t u(s)\right|^{(2r-k+1)/r} \, dx\right]^{\frac{2r}{r+1}} \\ &\quad + \varepsilon^{-2k_0} \left[\int_{\Omega} a(x) g(\partial_t u) \partial_t u \, dx\right]^{\frac{2}{r+1}}.\end{aligned} \tag{3.10}$$

Since $\frac{2r-k+1}{r} \leqslant \frac{2n}{n-2}$, from Sobolev's imbedding theorem, we have

$$\int_{\Omega_2} a^{2/p} \left|\partial_t u(s)\right|^2 \, dx \leqslant \varepsilon^{2k_0} \left\|(u_0, u_1)\right\|_{X^1(\Omega)}^{2\alpha} + \varepsilon^{-2k_0} \left[\int_{\Omega} a(x) g(\partial_t u) \partial_t u \, dx\right]^{\frac{2}{r+1}}, \tag{3.11}$$

where

$$\alpha = 1 + \frac{r-k}{r+1}. \tag{3.12}$$

Collecting (3.7), (3.8) and (3.11), and integrating with respect to s from t to $T+t$, we obtain the following estimate:

$$\int_t^{T+t}\int_{\Omega_2} \left|a(x)g(\partial_t u)\right|^2 \,dx\,ds \;\leqslant\; C\varepsilon^{k_0}\,\|(u_0,u_1)\|^{2\alpha}_{X^1(\Omega)} + C\varepsilon^{k_0'}\,\|(u_0,u_1)\|^{2p+\sigma}_{X^1} + C\varepsilon^{-3k_0}\left[\int_t^{T+t}\int_\Omega a(x)g(\partial_t u)\partial_t u\,dx\,ds\right]^{\frac{2}{r+1}} \tag{3.13}$$

Selecting in (3.13) $\varepsilon := \varepsilon^{k_0}$, we obtain (3.1). □

Lemma 3.2. *There exists $C>0$, such that the following estimate*

$$\int_t^{t+T}\int_{\Omega_1} \left|a(x)g(\partial_t u)\right|^2 \,dx\,ds \leqslant C\left[\int_t^{t+T}\int_\Omega a(x)g(\partial_t u)\partial_t u\,dx\,ds\right]^{\frac{2}{r+1}} \tag{3.14}$$

holds for any $t \geqslant 0$.

Proof. From (1.7) we obtain the following estimates:

$$\begin{aligned}\int_{\Omega_1} \left|a(x)g(\partial_t u)\right|^2\,dx &\leqslant C\int_{\Omega_1} a^2(x)(g(\partial_t u)\partial_t u)^{2/r+1}\,dx\\ &\leqslant C\int_{\Omega_1} a(x)^{2-2/r+1}(a(x)g(\partial_t u)\partial_t u)^{2/r+1}\,dx\\ &\leqslant C\left[\int_\Omega a(x)g(\partial_t u)\partial_t u\,dx\right]^{\frac{2}{r+1}}.\end{aligned} \tag{3.15}$$

Consequently, if we integrate with respect to s from t to $T+t$, we find by Holder's inequality (3.14). This completes the proof of the lemma. □

Lemma 3.3. *Let u be a solution to (1.9) starting from $(u_0,u_1) \in X^1(\Omega)$. Then there exist $\mu > 0$ and $\lambda_0 > 0$ such that for any $\lambda \geqslant \lambda_0$ the following estimate*

$$E(u,T+t) \leqslant C\left[\frac{1}{\lambda}\Phi\big(\,\|(u_0,u_1)\|^2_{X^1(\Omega)}\big) + e^{\mu\lambda}\Big(\int_t^{T+t}\int_\Omega a(x)g(\partial_t u)\partial_t u\,dx\,ds\Big)^{\frac{2}{r+1}}\right] \tag{3.16}$$

holds for any $t \geqslant 0$. *In the above*

$$\Phi(z) = \begin{cases} 1 + z + z^{1+\frac{r-k}{r+1}} & \textit{if} \quad p = 1, \\ 1 + z + z^{1+\frac{r-k}{r+1}} + z^{p+\frac{\sigma}{2}} & \textit{if} \quad p > 1. \end{cases} \tag{3.17}$$

Proof. Combining (3.1), (3.14), (2.5) and selecting $\varepsilon = e^{-2\mu\lambda}$ we obtain, for some constant $\mu_1 > 0$

$$E(u, T+t) \leqslant C \left[\frac{1}{\lambda} \Phi\big(\|(u_0, u_1)\|^2_{X^1} \big) + e^{\mu_1 \lambda} \left(\left(\int_t^{T+t} \int_\Omega g(\partial_t u) \partial_t u \, dx \, ds \right)^{\frac{2}{r+1}} + \int_t^{T+t} \int_\Omega a(x) \left| \partial_t u(s) \right|^2 dx \, ds \right) \right]. \tag{3.18}$$

To accomplish the proof of the lemma, we estimate the last term in RHS from (3.18).

First, using (1.8) we get for some $\delta = \frac{1}{r+1}$

$$\begin{aligned} \int_{\Omega_2} a(x) \left| \partial_t u(s) \right|^2 dx &\leqslant C \int_{\Omega_2} a^{1-\delta}(x) \left| \partial_t u(s) \right|^{2-\delta(k+1)} (a(x) g(\partial_t u) \partial_t u)^\delta \, dx \\ &\leqslant C \left[\int_\Omega \left| \partial_t u(s) \right|^{2-\delta(k+1)/1-\delta} dx \right]^{1-\delta} \\ &\quad \times \left[\int_\Omega a g(\partial_t u) \partial_t u \, dx \right]^\delta . \end{aligned} \tag{3.19}$$

By Sobolev's imbedding theorem, we deduce

$$\begin{aligned} \int_t^{T+t} \int_{\Omega_2} a(x) \left| \partial_t u(s) \right|^2 dx \, ds &\leqslant C \varepsilon \Phi(\|(u_0, u_1)\|^2_{X^1}) \\ &\quad + \varepsilon^{-1} \Big(\int_t^{T+t} \int_\Omega a(x) g(\partial_t u) \partial_t u \, dx \, ds \Big)^{\frac{2}{r+1}} . \end{aligned} \tag{3.20}$$

On the other hand, using (1.7), we obtain

$$\begin{aligned} \int_{\Omega_1} a(x) \left| \partial_t u(s) \right|^2 dx &\leqslant C \int_{\Omega_1} \Big(a(x) g(\partial_t u) \partial_t u \Big)^{\frac{2}{r+1}} dx \\ &\leqslant C \left[\int_\Omega a(x) g(\partial_t u) \partial_t u \, dx \right]^{\frac{2}{r+1}} . \end{aligned} \tag{3.21}$$

Combining (3.21), (3.18) and selecting $\varepsilon = e^{-2\mu_1\lambda}$, we obtain

$$E(u, T+t) \leqslant C \left[\frac{1}{\lambda} \Phi\Big(\|(u_0, u_1)\|^2_{X^1(\Omega)} \Big) + e^{\mu'\lambda} \left(\int_t^{T+t} \int_\Omega a(x) g(\partial_t u) \partial_t u \, dx \right)^{\frac{2}{r+1}} \right] \tag{3.22}$$

which completes the proof. □

We now turn to the proof of the first main result of the paper.

3.2. *End of the proof of Theorem 2*

Let $t \geqslant 0$. By Lemma 3.3, we have

$$E(u, T+t) \leqslant C \left[\frac{1}{\lambda} \Phi(\|(u_0, u_1)\|^2_{X^1}) + e^{\mu\lambda} \Big(E(u, t) - E(u, T+t) \Big)^{\frac{2}{r+1}} \right]. \tag{3.23}$$

For $t = nT$ and by setting $b_n = E(u, nT)$, we obtain

$$b_{n+1} \leqslant \frac{C_0}{\lambda} \Phi + e^{\mu\lambda} \Big(b_n - b_{n+1} \Big)^{\frac{2}{r+1}}. \tag{3.24}$$

Since $\Phi > 1$ we set $a_n = b_n . \Phi^{-1}$, and we obtain

$$a_{n+1} \leqslant \frac{C_0}{\lambda} + e^{\mu\lambda} \Big(a_n - a_{n+1} \Big)^{\frac{2}{r+1}}. \tag{3.25}$$

Applying Lemma 2.2 with $\varepsilon_1 = \varepsilon_2 = \frac{2}{r+1}$ and $\eta_1 = \eta_2 = 0$, we get

$$b_n \leqslant C \Phi(\log(n))^{-1}, \quad n \geqslant 2. \tag{3.26}$$

This completes the proof of Theorem 2.

4. Proof of the weak observation

We will now prove the linear estimates (2.4). This will be done in terms of the FBI transformation.

4.1. Preliminary and elliptic estimation

This section contains preliminary material, needed to prove the important inequality (2.4) for the solutions of (2.2). Denote for $T > 0$ and $\tilde{\omega} \subset\subset \omega$

$$\Omega_T =]-T, T[\times\Omega, \quad \tilde{\omega}_T =]-T, T[\times\tilde{\omega}. \tag{4.1}$$

We introduce the partial FBI transformation T_λ. It is defined for $u \in \mathcal{S}(\mathbb{R}^{n+1})$, the space of rapidly decreasing functions, by

$$T_\lambda u(z,x) = \sqrt{\frac{\lambda}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\lambda}{2}(z-y)^2} u(y,x)\,dy, \quad z = t + is. \tag{4.2}$$

Then we have the following estimate (see [22]):

$$\left|D_x^\alpha T_\lambda u(z,x)\right| \leqslant C\sqrt{\frac{\lambda}{2\pi}} e^{\lambda s^2} e^{-\frac{\lambda}{2}[d(t,supp(u))]^2} \sup \left\| D_x^\alpha u(.,x)\right\|_2^2, \tag{4.3}$$

for any $u \in C_0^\infty(\mathbb{R}\times\mathbb{R}^n)$.

In the sequel, we assume that T is large and $s \in [-1,1]$, $t \in]\frac{T}{3}, \frac{2T}{3}[$. We introduce a cut-off function χ satisfying $0 \leqslant \chi \leqslant 1$, $\chi(t) \in \mathcal{C}_0^\infty(\mathbb{R})$ and

$$\chi(t) = \begin{cases} 1, & t \in [2, T-2], \\ 0, & t \in [0,1] \cup [T-1, T]. \end{cases} \tag{4.4}$$

Let $\tilde{v}$ be a solution of the following linear conservative wave equation:

$$\begin{cases} \partial_t^2 \tilde{v} - \Delta \tilde{v} = 0 & \text{in} \quad \Omega \times \mathbb{R}, \\ \tilde{v} = 0 & \text{on} \quad \Gamma \times \mathbb{R}, \\ \tilde{v}(x,0) = u_0, \quad \partial_t \tilde{v}(x,0) = u_1 & \text{in} \quad \Omega. \end{cases} \tag{4.5}$$

We set $v = \chi\tilde{v}$.

In connection with the operator $\partial_t^2 - \Delta$, we define an elliptic operator by

$$Q = \partial_s^2 + \Delta. \tag{4.6}$$

Lemma 4.1. *There exists $C, \mu, \lambda_0 > 0$ such that for any $\lambda \geqslant \lambda_0$ the following estimate*

$$\|T_\lambda v\|^2_{H^1(\Omega_{1/2})} \leqslant C\left(e^{-\frac{\lambda}{\mu}} \|v\|^2_{H^1(\Omega_T)} + e^{\mu\lambda} \|v\|^2_{L^2(\omega_T)}\right) \tag{4.7}$$

holds for any $v = \chi\tilde{v}$ where $\tilde{v}$ is a solution of (4.5).

Proof. In order to prove (4.7), we invoke the following interpolation inequality proved by Lebeau-Robbiano [13]. From [13], there exists $0 < \nu < 1$ such that

$$\|w\|_{H^1(\Omega_{1/2})} \leqslant C\, \|w\|^{\nu}_{H^1(\Omega_1)} \left(\|Qw\|_{L^2(\Omega_1)} + \|w\|_{H^1(\tilde{\omega}_{1/2})} \right)^{1-\nu}, \tag{4.8}$$

for any $w \in H^2(\Omega_1)$, and such that $w(t,x) = 0$ on $\partial\Omega_1$. We define a cut-off function $\tilde{\chi}$ such that $\tilde{\chi}(s,x) = 1$ near $\tilde{\omega}_{1/2}$ and is supported in ω_1. Then we have

$$(1+Q)(\tilde{\chi} w) = \tilde{\chi} Q w + [Q, \tilde{\chi}]w + \tilde{\chi} w. \tag{4.9}$$

Therefore

$$\begin{aligned} \|w\|_{H^1(\tilde{\omega}_{1/2})} &\leqslant \|(1+Q)(\chi w)\|_{H^{-1}} + \|w\|_{L^2(\omega_1)} \\ &\leqslant C\Big(\|Qw\|_{L^2(\Omega_1)} + \|w\|_{L^2(\omega_1)} \Big). \end{aligned} \tag{4.10}$$

Combining (4.10) and (4.8), we obtain

$$\|w\|_{H^1(\Omega_{1/2})} \leqslant C\, \|w\|^{\nu}_{H^1(\Omega_1)} \left(\|Qw\|_{L^2(\Omega_1)} + \|w\|_{L^2(\omega_1)} \right)^{1-\nu}. \tag{4.11}$$

Denote $w_{\lambda,t}(s,x) = T_\lambda v(z,x)$. Then we get

$$\begin{aligned} Q w_{\lambda,t}(s,x) &= (\partial_s^2 + \Delta)(T_\lambda v(t+is,x)) \\ &= \sqrt{\frac{\lambda}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\lambda}{2}(z-y)^2} \big[2(D_t\chi)(D_t v) + v(D_t^2\chi)\big]\, dy \\ &= T_\lambda f(z,x), \end{aligned} \tag{4.12}$$

where f is given by

$$f = -\big[2(D_t\chi)(D_t v) + v(\chi'')\big]. \tag{4.13}$$

Since χ' and χ'' are supported in $[0,2] \cup [T-2, T]$, by (4.3), we obtain

$$\left\| Q w_{\lambda,t} \right\|_{L^2(\Omega_1)} \leqslant e^{-\beta T \lambda}\, \|v\|_{H^1(\Omega_T)}, \tag{4.14}$$

for some $\beta > 0$. On the other hand we have for some constant α independent of T

$$\|T_\lambda v\|_{H^1(\Omega_1)} \leqslant Ce^{\alpha\lambda} \|v\|_{H^1(\Omega_T)} \tag{4.15}$$

which combined with (4.14) and (4.8) yields

$$\|T_\lambda v\|_{H^1(\Omega_{1/2})} \leqslant Ce^{\alpha\lambda} \|v\|^{\nu}_{H^1(\Omega_T)} \left[e^{-\beta T\lambda} \|v\|_{H^1(\Omega_T)} + e^{\alpha\lambda} \|v\|_{L^2(\omega_T)}\right]^{1-\nu}. \tag{4.16}$$

We easily obtain

$$\|T_\lambda v\|_{H^1(\Omega_{1/2})} \leqslant \varepsilon^{k_0} e^{\alpha'\lambda} \|v\|^2_{H^1(\Omega_T)} + \varepsilon^{-k_0'} \left[e^{-\beta T\lambda} \|v\|^2_{H^1(\Omega_T)} + e^{\alpha\lambda} \|v\|^2_{L^2(\omega_T)}\right], \tag{4.17}$$

where $k_0 = \frac{1}{\nu}$; $k_0' = \frac{1}{1-\nu}$.

Finally selecting $\varepsilon = e^{-\frac{\alpha'+\beta T}{k_0+k_0'}\lambda}$ with T sufficiently large, we obtain

$$\|T_\lambda v\|^2_{H^1(\Omega_{1/2})} \leqslant e^{-\mu\lambda} \|v\|^2_{H^1(\Omega_T)} + e^{\mu'\lambda} \|v\|^2_{L^2(\omega_T)}. \tag{4.18}$$

This completes the proof. $\square$

4.2. Proof of Proposition 2.1

We shall begin with the following lemma.

Lemma 4.2. *There exists $C, \mu, \lambda_0 > 0$, such that for any $\lambda \geqslant \lambda_0$, we have*

$$\|v\|^2_{L^2(\Omega_T)} \leqslant \frac{C}{\lambda} \|v\|^2_{H^1(\Omega_T)} + Ce^{\mu\lambda} \|v\|^2_{L^2(\omega_T)}, \tag{4.19}$$

for any $v = \chi\tilde{v}$ where $\tilde{v}$ is a solution of (4.5).

Proof. If we take $s = 0$ in (4.2) $(z = t)$, we get

$$\begin{aligned} T_\lambda u(t,x) &= \sqrt{\frac{\lambda}{2\pi}} \int_{\mathbb{R}} e^{-\frac{\lambda}{2}(t-y)^2} u(y,x)\, dy \\ &= (G_\lambda * u)(t,x), \end{aligned} \tag{4.20}$$

where

$$G_\lambda(t) = \sqrt{\frac{\lambda}{2\pi}}\, e^{-\frac{\lambda}{2}t^2}. \tag{4.21}$$

Then we have

$$\widehat{v}(\tau, x) - \widehat{T_\lambda v}(\tau, x) = (1 - \widehat{G_\lambda})\widehat{v}(\tau, x). \tag{4.22}$$

Furthermore, we can immediately verify that

$$\left|1 - \widehat{G_\lambda}\right| = \left|1 - e^{-\frac{\tau^2}{\lambda}}\right| \leqslant \frac{\tau}{\sqrt{\lambda}}. \tag{4.23}$$

Therefore

$$\|v - T_\lambda v\|_{L^2}^2 = \left\|v - w_{\lambda,t}(0, x)\right\|^2 \leqslant \frac{C}{\lambda} \|v\|_{H^1(\Omega_T)}^2 \tag{4.24}$$

which easily implies that

$$\begin{aligned} \|v\|_{L^2(\Omega_T)}^2 &\leqslant C\left[\left\|v - w_{\lambda,t}(0, x)\right\|^2 + \left\|w_{\lambda,t}(0, x)\right\|_{L^2_{t,x}}^2\right] \\ &\leqslant C\left[\frac{1}{\lambda}\|v\|_{H^1(\Omega_T)}^2 + \left\|w_{\lambda,t}(0, .)\right\|_{L^2_{t,x}}^2\right]. \end{aligned} \tag{4.25}$$

By the Cauchy formula (see [21]) and (4.7), we obtain

$$\left\|w_{\lambda,t}(0, .)\right\|_{L^2_{t,x}}^2 \leqslant e^{-\mu\lambda} \|v\|_{H^1(\Omega_T)}^2 + e^{\mu'\lambda} \|v\|_{L^2(\omega_T)}^2 . \tag{4.26}$$

This complete the proof of (4.19). $\square$

We now turn to the proof of Proposition 2.1. Applying Lemma 4.2 to $\partial_t \tilde{v}$, we obtain

$$\int_{T/3}^{2T/3} \int_\Omega \left|\partial_t \tilde{v}(t, x)\right|^2 dx\, dt \leqslant \frac{C}{\lambda} \|v\|_{H^2(\Omega_T)}^2 + Ce^{\mu\lambda} \left\|\partial_t v\right\|_{L^2(\omega_T)}^2 . \tag{4.27}$$

On the other hand, multiplying the first equation in (4.5) by $\tilde{v}$ and integrating by part with respect to t, one easily derives the estimation

$$\int_{T/3}^{2T/3} \int_\Omega |\nabla \tilde{v}(t, x)|^2 dx\, dt \leqslant \int_{T/3}^{2T/3} \int_\Omega \left|\partial_t \tilde{v}(t, x)\right|^2 dx\, dt + 2E(u_0, u_1). \tag{4.28}$$

Collecting (4.28) and (4.27), we obtain

$$TE(u_0,u_1)\leqslant\left[\frac{C}{\lambda}\|v\|^2_{H^2(\Omega_T)}+Ce^{\mu\lambda}\left\|\partial_t v\right\|^2_{L^2(\omega_T)}\right]+CE(u_0,u_1). \tag{4.29}$$

Selecting T sufficiently large, we obtain (2.4).

5. The non-autonomous case

In this section we first recall the main result concerning the existence, uniqueness and regularity of solutions to the initial value problem associated with (1.23). We need the following preliminaries, which are essentially known although most of them are not explicitly listed in the literature. In the proof of the main result, we shall make use of the following estimates which will be used for the proof of Theorem 3:

$$\left\|\Delta u(t)\right\|^2+\left\|\nabla\partial_t u(t)\right\|^2\leqslant C_0\Phi(\|(u_0,u_1)\|^2_{X^1(\Omega)}) \tag{5.1}$$

for any $t\geqslant 0$ and u solution of (1.23) starting from $(u_0,u_1)\in X^1(\Omega)$.

The existence and uniqueness part of the theorem is standard (e.g., Lions and Strauss [16], Nakao [18]) and for the proof of the theorem it suffices to derive estimate (5.1) for the assumed smooth solution.

5.1. The strong solution

Our aim is to find suitable a priori estimates ensuring the existence and uniqueness of the solution in the class mentioned above. To this end, we start with standard energy identities

$$E(u,t)+\int_0^t\int_\Omega\sigma(s)a(x)g(\partial_t u)\partial_t u\,dx\,ds=E(u,0). \tag{5.2}$$

Next, we assume that g is a smooth function. Differentiating the equation with respect to t, we have for $v=\partial_t u$

$$\partial_t^2 v-\Delta v+\sigma'(t)a(x)g(v)+\sigma(t)a(x)g'(v)\partial_t v=0. \tag{5.3}$$

Multiplying by $\partial_t v$ and integrating from 0 to t, gives

$$\begin{aligned}E(v,t)&+\int_0^t\int_\Omega\sigma(s)a(x)g'(v)\left|\partial_t v\right|^2\,dx\,ds\\&=E(v,0)-\int_0^t\int_\Omega\sigma'(s)a(x)g(v)\partial_t v\,dx\,ds.\end{aligned} \tag{5.4}$$

On the other hand, by virtue of (1.23), (1.7), (1.8) and Sobolev's imbedding theorem, we obtain

$$\begin{aligned} E(v,0) &= \left\|\partial_t^2 u(0)\right\|_2^2 + \left\|\nabla\partial_t u(0)\right\|_2^2 \\ &\leqslant C\left[\left\|\Delta u_0\right\|_2^2 + \|\nabla u_1\|_2^2 + \int_{\Omega_1} |u_1(x)|^{\frac{2}{r}}\, dx + \int_{\Omega_2} |u_1(x)|^2\, dx\right] \\ &\leqslant C(\left\|\Delta u_0\right\|_2^2 + \|\nabla u_1\|_2^2 + \|u_1\|_2^{2/r} + \|\nabla u_1\|_2^2) \leqslant C\Phi(\|u_0,u_1\|_{X^1}^2). \end{aligned} \tag{5.5}$$

Combining (5.5) with (5.4), we deduce the following basic inequality:

$$E(v,t) \leqslant C\Phi(\|u_0,u_1\|_{X^1}^2) + \int_0^t \int_\Omega \left|\sigma'(s)a(x)g(v)\partial_t v\right|\, dx\, ds. \tag{5.6}$$

If $r=1$, the right-hand side is further estimated by means of (1.7) and (1.8)

$$\begin{aligned} \int_0^t \int_\Omega \left|\sigma'(s)a(x)g(v)\partial_t v\right|\, dx\, ds &\leqslant C\int_0^t \int_\Omega \left|\sigma'(s)a(x)v(x,s).\partial_t v(x,s)\right|\, dx\, ds \\ &\leqslant C\int_0^t \int_\Omega \sigma(s)a(x)\,|v(x,s)|^2\, dx\, ds \\ &\quad + \int_0^t \int_\Omega \frac{\sigma'(s)^2}{\sigma(s)}\left|\partial_t v(x,s)\right|^2\, dx\, ds. \end{aligned} \tag{5.7}$$

Using (5.2), (1.7) and (1.8), we obtain

$$\int_0^t \int_\Omega \left|\sigma'(s)a(x)g(v)\partial_t v\right|\, dx\, ds \leqslant CE(u,0) + \int_0^t \frac{\sigma'(s)^2}{\sigma(s)} E(v,s)\, ds. \tag{5.8}$$

Collecting (5.8) and (5.6) by Gronwall's inequality and the fact that $\theta < 0$, we find that

$$\left\|\partial_t^2 u(t)\right\|^2 + \left\|\partial_t \nabla u(t)\right\|^2 \leqslant C\Phi\Big(\|(u_0,u_1)\|_{X^1}^2\Big). \tag{5.9}$$

Next, using the equation, we obtain

$$\begin{aligned} \left\|\Delta u(t)\right\|^2 &= \left\|\partial_t^2 u(t) + \sigma(t)ag(\partial_t u)\right\|^2 \\ &\leqslant C(\left\|\partial_t^2 u(t)\right\|^2 + \left\|\partial_t u(t)\right\|^2) \\ &\leqslant C\Phi(\|(u_0,u_1)\|_{X^1}^2). \end{aligned} \tag{5.10}$$

The uniqueness and existence part of the lemma follows from standard estimates (5.10) and (5.9) (see Lions and Straus [16]).

Now we assume that $r > 1$.

Let $\alpha \in \mathbb{R}$ the right-hand side of (5.4) be estimated by

$$\int_0^t \int_\Omega \left|\sigma'(s)a(x)g(v)\partial_t v\right| \,dx\,ds \leqslant \int_0^t \int_\Omega \sigma^\alpha(s)a(x)\,|g(v)|^2 \,dx\,ds + C\int_0^t \int_\Omega \frac{\sigma'(s)^2}{\sigma^\alpha(s)} \left|\partial_t v\right|^2 \,dx\,ds. \tag{5.11}$$

Taking into account (1.7) and (1.8), we obtain

$$\begin{aligned}\int_0^t \int_\Omega \sigma^\alpha(s)a(x)\,|g(v)|^2 \,dx\,ds &\leqslant \int_0^t \int_{\Omega_1} \sigma^\alpha(s)a(x)(g(v).v)^{\frac{2}{r+1}} \,dx\,ds \\ &\quad + \int_0^t \int_{\Omega_2} \sigma^\alpha(s)a(x)(g(v).v)\,dx\,ds \\ &\leqslant C\int_0^t \sigma(s)^{\alpha-\frac{2}{r+1}} \Big(\int_\Omega (a(x)\sigma(s)g(v)v\,dx\Big)^{\frac{2}{r+1}} ds \\ &\quad + \int_0^t \sigma(s)^{\alpha-1}\Big(\int_\Omega a(x)\sigma(s)g(v)v\,dx\Big)\,ds \\ &= I_1(t) + I_2(t).\end{aligned} \tag{5.12}$$

Using Hölder inequality, we obtain

$$\begin{aligned}I_1(t) &\leqslant C\left[\int_0^t \sigma(s)^{\frac{\alpha(r+1)-2}{r-1}}\,ds\right]^{\frac{r-1}{r+1}} \left[\int_0^t \int_\Omega a(x)\sigma(s)g(v)v\,dx\,ds\right]^{\frac{2}{r+1}} \\ &\leqslant CE(u_0,u_1)^{\frac{2}{r+1}} \left[\int_0^t \sigma(s)^{\frac{\alpha(r+1)-2}{r-1}}\,ds\right]^{\frac{r-1}{r+1}} \\ &\leqslant C\Phi(\|u_0,u_1\|^2_{X^1}) \left[\int_0^t \sigma(s)^{\frac{\alpha(r+1)-2}{r-1}}\,ds\right]^{\frac{r-1}{r+1}}.\end{aligned} \tag{5.13}$$

By a simple calculation, we obtain

$$\frac{1}{r+1}\left(2 - \frac{r-1}{\theta}\right) < 2 - \frac{1}{\theta}. \tag{5.14}$$

Selecting now α such that

$$\frac{1}{r+1}\left(2-\frac{r-1}{\theta}\right) < \alpha < 2-\frac{1}{\theta} \tag{5.15}$$

we obtain $r-1 < -\theta(\alpha(r+1)-2)$, and from (5.13) we get

$$I_1(t) \leqslant C_0\Phi(\|(u_0,u_1)\|^2_{X^1}). \tag{5.16}$$

Further, using $1 \leqslant \frac{1}{r+1}(2-\frac{r-1}{\theta}) < \alpha$ and $-1 < \theta < 0$, we get

$$I_2(t) \leqslant C_0\Phi(\|(u_0,u_1)\|^2_{X^1}). \tag{5.17}$$

Collecting (5.17) and (5.16), we obtain

$$E(v,t) \leqslant C\Phi(\|(u_0,u_1)\|^2_{X^1}) + \int_0^t \frac{\sigma'(s)^2}{\sigma(s)^\alpha} E(v,s)\,ds. \tag{5.18}$$

Using the second inequality of (5.15) and applying the Gronwall lemma, we obtain

$$E(v,t) \leqslant C\Phi(\|(u_0,u_1)\|^2_{X^1}), \quad \forall t \geqslant 0. \tag{5.19}$$

On the other hand using the equation, we obtain

$$\begin{aligned}
\left\|\Delta u(t)\right\|^2 &\leqslant C\left(\left\|\partial_t^2 u(t)\right\|^2 + \int_\Omega |\sigma(t)a(x)g(v)|^2\,dx\right)\\
&\leqslant C\left[\Phi(\|(u_0,u_1)\|^2_{X^1}) + \int_{\Omega_1} a(x)\sigma(t)^2\,|g(v)|^2\,dx + E(u,0)\right]\\
&\leqslant C\left[\Phi(\|(u_0,u_1)\|^2_{X^1}) + \int_{\Omega} |v(x,t)|^{\frac{2}{r}}\,dx + E(u,0)\right]\\
&\leqslant C\Phi\Big(\,\|u_0,u_1\|^2_{X^1}\Big).
\end{aligned} \tag{5.20}$$

Collecting (5.20) and (5.19), we obtain (5.1).

5.2. Proof of Theorem 3

Lemma 5.1. *Let u be a solution to (1.23) starting from $(u_0,u_1) \in X^1(\Omega)$ and let $T > 0$ be an arbitrary constant. Then there exists a constant $C > 0$ such that the* *following estimates*

$$\int_t^{T+t}\int_\Omega \left|b(s,x)g(\partial_t u)\right|^2 \, dx\, ds \leqslant C\Big(E(u,t)-E(u,T+t)\Big)^{\frac{2}{r+1}} \tag{5.21}$$

and

$$\begin{aligned}\int_t^{T+t}\int_\Omega a(x)\left|\partial_t u\right|^2 \, dx\, ds \;&\leqslant\; C\Big[\varepsilon\,\Phi(\|(u_0,u_1)\|^2_{X^1}) \\ &\quad + \varepsilon^{-1}(1+t)^{\frac{2|\theta|}{r+1}}(E(u,t)-E(u,T+t))^{\frac{2}{r+1}}\Big]\end{aligned} \tag{5.22}$$

hold for any $t\geqslant t_0$ *and* $\varepsilon>0$.

Proof. From (1.7), we obtain the following estimate:

$$\begin{aligned}\int_{\Omega_1}\left|b(s,x)g(\partial_t u)\right|^2 \, dx \;&\leqslant\; C\int_{\Omega_1} b^2(s,x)(g(\partial_t u)\partial_t u)^{\frac{2}{r+1}}\, dx \\ &\leqslant\; C\int_\Omega \sigma(s)^{2-\frac{2}{r+1}}(\sigma(s)a(x)g(\partial_t u)\partial_t u)^{\frac{2}{r+1}}\, dx \\ &\leqslant\; C\sigma(s)^{\frac{2r}{r+1}}\Big(\int_\Omega \sigma(s)a(x)g(\partial_t u)\partial_t u\, dx\Big)^{\frac{2}{r+1}},\end{aligned} \tag{5.23}$$

which further implies

$$\begin{aligned}\int_t^{T+t}\int_{\Omega_1}\left|b(s,x)g(\partial_t u)\right|^2 \, dx\, ds \;&\leqslant\; C\int_t^{T+t}\sigma(s)^{\frac{2r}{r+1}}\Big(\int_\Omega \sigma(s)a(x)g(\partial_t u)\partial_t u\, dx\Big)^{\frac{2}{r+1}} ds \\ &\leqslant\; C\Big(\int_t^{T+t}\int_\Omega b(s,x)g(\partial_t u)\partial_t u\, dx\, ds\Big)^{\frac{2}{r+1}} \\ &\leqslant\; C\Big(E(u,t)-E(u,T+t)\Big)^{\frac{2}{r+1}}.\end{aligned} \tag{5.24}$$

On the other hand, using the growth condition imposed on g by (1.8) (with $p=1$), we obtain

$$\begin{aligned}\int_t^{T+t}\int_{\Omega_2}\left|b(s,x)g(\partial_t u)\right|^2 \, dx\, ds \;&\leqslant\; C\int_t^{T+t}\int_\Omega \sigma^2(s)a(x)g(\partial_t u)\partial_t u\, dx\, ds \\ &\leqslant\; C\Big(E(u,t)-E(u,T+t)\Big).\end{aligned} \tag{5.25}$$

Finally, we remark that

$$\lim_{t\to\infty}(E(u,t)-E(u,T+t))=0. \tag{5.26}$$

This completes the proof of (5.21) for $t\geqslant t_0$.

Our next step is to prove (5.22). Using condition (1.7), we get

$$\int_{\Omega_1} a(x)\left|\partial_t u(s)\right|^2\,dx \leqslant C\Big(\int_{\Omega} a(x)g(\partial_t u)\partial_t u\,dx\Big)^{\frac{2}{r+1}}. \tag{5.27}$$

Then we have

$$\begin{aligned}\int_t^{T+t}\!\!\int_{\Omega_1} a(x)\left|\partial_t u\right|^2\,dx\,ds &\leqslant C\int_t^{T+t}\sigma(s)^{-\frac{2}{r+1}}\Big(\int_{\Omega}\sigma(s)a(x)g(\partial_t u)\partial_t u\,dx\Big)^{\frac{2}{r+1}}\,ds\\ &\leqslant C(1+t)^{2\frac{|\theta|}{r+1}}\Big(E(u,t)-E(u,T+t)\Big)^{\frac{2}{r+1}}.\end{aligned} \tag{5.28}$$

Similarly using assumption (1.8), we obtain for $\delta=\frac{1}{r+1}$

$$\begin{aligned}\int_{\Omega_2} a(x)\left|\partial_t u(s)\right|^2\,dx &\leqslant C\int_{\Omega_2} a^{1-\delta}(x)\left|\partial_t u(s)\right|^{2-\delta(k+1)}(a(x)g(\partial_t u)\partial_t u)^{\delta}\,dx\\ &\leqslant C\left[\int_{\Omega}\left|\partial_t u(s)\right|^{2-\delta(k+1)/1-\delta}\,dx\right]^{1-\delta}\\ &\quad\times\left[\int_{\Omega}a(x)g(\partial_t u)\partial_t u\,dx\right]^{\delta}\leqslant C\varepsilon\,\Phi(\|(u_0,u_1)\|^2_{X^1})\\ &\quad+\varepsilon^{-1}\Big(\int_{\Omega}a(x)g(\partial_t u)\partial_t u\,dx\Big)^{\frac{2}{r+1}}.\end{aligned} \tag{5.29}$$

Then we can conclude that

$$\begin{aligned}\int_t^{T+t}\!\!\int_{\Omega_2} a(x)\left|\partial_t u\right|^2\,dx\,ds &\leqslant C\varepsilon\,\Phi\Big(\,\|(u_0,u_1)\|^2_{X^1}\Big)+C\varepsilon^{-1}(1+t)^{2\frac{|\theta|}{r+1}}\\ &\quad\times\Big(\int_t^{T+t}\!\!\int_{\Omega}\sigma(s)a(x)g(\partial_t u)\partial_t u\,dx\,ds\Big)^{\frac{2}{r+1}}\\ &\leqslant C\varepsilon\,\Phi\Big(\,\|(u_0,u_1)\|^2_{X^1}\Big)\\ &\quad+C\varepsilon^{-1}(1+t)^{2\frac{|\theta|}{r+1}}\Big(E(u,t)-E(u,T+t)\Big)^{\frac{2}{r+1}}.\end{aligned} \tag{5.30}$$

Collecting (5.30) and (5.28), we obtain (5.22). □

Lemma 5.2. *Let u be a solution to (1.23) starting from $(u_0, u_1) \in X^1(\Omega)$. Then there exists $\mu > 0$ and $\lambda_0 > 0$ such that for any $\lambda \geqslant \lambda_0$ we have the following estimate*:

$$E(u, T+t) \leqslant \frac{C}{\lambda}\Phi\Big(\|(u_0,u_1)\|^2_{X^1(\Omega)}\Big) + Ce^{\mu\lambda}\left[(1+t)^{2\frac{|\theta|}{r+1}}\Big(E(u,t)-E(u,t+T)\Big)^{\frac{2}{r+1}}\right], \quad (5.31)$$

where Φ is a real function defined by

$$\Phi(z) = 1 + z + z^{1+\frac{r-k}{r+1}}. \quad (5.32)$$

Proof. Let $t \geqslant 0$. We set $\phi(s,x) = u(t+s,x)$. Then ϕ satisfies

$$\begin{cases} \partial_s^2\phi - \Delta\phi + b_t(s,x)g(\partial_s\phi) = 0 & \text{in} \quad \Omega \times (0,\infty), \\ \phi = 0 & \text{on} \quad \partial\Omega \times (0,\infty), \\ \phi(x,0) = u(t,x), \quad \partial_s\phi(x,0) = \partial_t u(t,x) & \text{in} \quad \Omega. \end{cases} \quad (5.33)$$

Therefore, using the decomposition argument of Section 2 and (5.1) we easily derive that

$$E(u,t+T) \leqslant C\Big(\frac{1}{\lambda}\Phi(\|(u_0,u_1)\|^2_{X^1(\Omega)}) + e^{\mu\lambda}\int_t^{t+T}\int_\Omega a(x)\left|\partial_t u(s,x)\right|^2 + \left|b(s,x)g(\partial_t u)\right|^2\,dx\,ds\Big). \quad (5.34)$$

Taking into account (5.21) and (5.22) and selecting $\varepsilon = e^{-2\mu\lambda}$, we obtain

$$E(u,t+T) \leqslant \frac{C}{\lambda}\Phi(\|(u_0,u_1)\|^2_{X^1(\Omega)}) + Ce^{\mu\lambda}\left[(1+t)^{2\frac{|\theta|}{r+1}}(E(u,t)-E(u,t+T))^{\frac{2}{r+1}}\right]. \quad (5.35)$$

We now turn to the proof of Theorem 3.

For $t = nT$, setting $b_n = E(u, n.T)$, we obtain

$$b_{n+1} \leqslant \frac{C_0}{\lambda}\Phi + e^{\mu\lambda}n^{2\frac{|\theta|}{r+1}}\Big(b_n - b_{n+1}\Big)^{\frac{2}{r+1}}, \quad (5.36)$$

such that $\Phi > 1$. We set $a_n = b_n\Phi^{-1}$. Then we obtain

$$a_{n+1} \leqslant \frac{C_0}{\lambda} + e^{\mu\lambda}n^{2\frac{|\theta|}{r+1}}\Big(a_n - a_{n+1}\Big)^{\frac{2}{r+1}}. \quad (5.37)$$

Applying Lemma 2.2 with $\varepsilon_1 = \varepsilon_2 = \frac{2}{r+1}$ and $\eta_1 = \eta_2 = \frac{2|\theta|}{r+1}$, we get

$$b_n \leqslant C\Phi(\log(n))^{-1}, \quad n \geqslant 2. \tag{5.38}$$

This completes the proof of Theorem 3. □

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